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Hyper-Hamiltonian generalized Petersen graphs

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Abstract

Assume that \( n \) and \( k \) are positive integers with \( n \geq 2k + 1 \). A non-Hamiltonian graph \( G \) is hypo-Hamiltonian if \( G - v \) is Hamiltonian for any \( v \in V(G) \). It is proved that the generalized Petersen graph \( P(n, k) \) is hypo-Hamiltonian if and only if \( k = 2 \) and \( n \equiv 5 \pmod{6} \). Similarly, a Hamiltonian graph \( G \) is hyper-Hamiltonian if \( G - v \) is Hamiltonian for any \( v \in V(G) \). In this paper, we will give some necessary conditions and some sufficient conditions for the hyper-Hamiltonian generalized Petersen graphs. In particular, \( P(n, k) \) is not hyper-Hamiltonian if \( n \) is even and \( k \) is odd. We also prove that \( P(3k, k) \) is hyper-Hamiltonian if and only if \( k \) is odd. Moreover, \( P(n, 3) \) is hyper-Hamiltonian if and only if \( n \) is odd and \( P(n, 4) \) is hyper-Hamiltonian if and only if \( n \neq 12 \). Furthermore, \( P(n, k) \) is hyper-Hamiltonian if \( k \) is even with \( k \geq 6 \) and \( n \geq 2k + 2 + (4k - 1)(4k + 1) \), and \( P(n, k) \) is hyper-Hamiltonian if \( k \geq 5 \) is odd and \( n \) is odd with \( n \geq 6k - 3 + 2k(6k - 2) \).

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1. Definitions and notations

For the graph terminology and notation we follow [1]. \( G \equiv (V, E) \) is a graph if \( V \) is a finite set and \( E \) is a subset of \( \{(u, v) \mid (u, v) \) is an unordered pair of \( V \} \). We say that \( V(G) \) is the vertex set and \( E(G) \) is the edge set of \( G \). We delimited a path \( P \) in a graph from a vertex \( v_0 \) to \( v_n \) by \( \langle v_0, v_1, v_2, \ldots, v_n \rangle \). A cycle is a path with at least three vertices such that its first vertex is the same as the last vertex. A cycle is a Hamiltonian cycle if it traverses every vertex of \( G \) exactly once. A graph is Hamiltonian if it has a Hamiltonian cycle.

Assume that \( n \) and \( k \) are positive integers with \( n \geq 2k + 1 \). We use \( \oplus \) to denote addition in integer modular \( n \). The generalized Petersen graph \( P(n, k) \) is the graph with vertex set \( \{i \mid 0 \leq i \leq n - 1\} \cup \{i' \mid 0 \leq i \leq n - 1\} \) and edge set \( \{(i, i + 1) \mid 0 \leq i \leq n - 1\} \cup \{(i, i') \mid 0 \leq i \leq n - 1\} \cup \{(i', (i \oplus k)' \mid 0 \leq i \leq n - 1\} \). The Hamiltonian generalized Petersen graph has been extensively studied [2–6]. In particular, Alspach [3] gave the classification on Hamiltonian generalized Petersen graphs as follows.

**Theorem 1** ([3]). \( P(n, k) \) is Hamiltonian if and only if it is neither (1) \( P(n, 2) \equiv \) \( P(n, n - 2) \equiv P(n, (n - 1)/2) \equiv P(n, (n + 1)/2) \), \( n \equiv 5 \pmod{6} \) nor (2) \( P(n, n/2) \), \( n \equiv 0 \pmod{4} \) and \( n \geq 8 \).
In this paper we concentrate on cubic generalized Petersen graphs, the case \( k = n/2 \) will be exclusive for consideration. Therefore \( P(n, k) \) is not Hamiltonian if and only if \( k = 2 \) and \( n \equiv 5 \pmod{6} \). However, all non-Hamiltonian generalized Petersen graphs satisfy another interesting property. A non-Hamiltonian graph \( G \) such that \( G - v \) is Hamiltonian for any \( v \in V(G) \) is called a hypo-Hamiltonian graph. It is proved that the set of hypo-Hamiltonian generalized Petersen graphs is actually the set of non-Hamiltonian generalized Petersen graphs [5].

Similarly, a Hamiltonian graph \( G \) is hyper-Hamiltonian if \( G - v \) is Hamiltonian for any \( v \in V(G) \). We are interested in the recognition of hyper-Hamiltonian generalized Petersen graphs. In [7], it is proved that \( P(n, 1) \) is hyper-Hamiltonian if and only if \( n \) is odd and \( P(n, 2) \) is hyper-Hamiltonian if and only if \( n \equiv 1, 3 \pmod{6} \).

In this paper, we will give some necessary conditions and some sufficient conditions for the hyper-Hamiltonian generalized Petersen graphs. In particular, \( P(n, k) \) is not hyper-Hamiltonian if \( n \) is even and \( k \) is odd. We also proved that \( P(3k, k) \) is hyper-Hamiltonian if and only if \( k \) is even. Moreover, \( P(n, 3) \) is hyper-Hamiltonian if and only if \( n \) is odd; \( P(n, 4) \) is hyper-Hamiltonian if and only if \( n \neq 12 \). Furthermore, \( P(n, k) \) is hyper-Hamiltonian if \( k \) is even with \( m \geq 6 \) and \( n \geq 2k + 2 + (4k - 1)(4k + 1) \), and \( P(n, k) \) is hyper-Hamiltonian if \( k \) is odd with \( m \geq 5 \) and \( n \) is odd with \( n \geq 6k - 3 + 2k(6k - 2) \).

Alspach et al. [2] proposed an interesting model, called lattice model, to describe a Hamiltonian cycle for given generalized Petersen graph. With the lattice model, a lattice diagram for a generalized Petersen graphs is a labeled graph in the \((x, y)\)-plane that possesses a closed or an open Eulerian trial. By appropriately interpreting the edges in the diagram, the Eulerian trail corresponds to a Hamiltonian cycle of a generalized Petersen graph. Based on this model, they proved that \( P(n, k) \) is Hamiltonian if \( k \geq 3 \) and \( n \) sufficiently large [2] and moreover classified the Hamiltonian generalized Petersen graphs [3].

The lattice model is described as follows. For more details, see [2]. In lattice model, a labeled lattice graph \( L(n, k) \) consists of lattice points in the \((x, y)\)-plane. If a lattice point \((a, b)\) is labeled with an integer \( i \) with \( 0 \leq i \leq n - 1 \), then \((a + 1, b)\) is labeled with \( i + 1 \) and \((a, b - 1)\) is labeled with \( i \oplus k \). A lattice diagram for \( P(n, k) \), denoted as \( D(n, k) \), is a subgraph of \( L(n, k) \) induced by the vertices with labels \( 0, 1, \ldots, n - 1 \) such that it possesses either a closed or an open Eulerian trail. A traversal of the Eulerian trail in \( D(n, k) \) obeys the following rules:

1. The trail does not change the direction when it passes through a vertex of degree 4.
2. Each label \( 0, 1, \ldots, n - 1 \) is encountered by the traversal once in a vertical direction and once in a horizontal direction.
3. If \( D(n, k) \) has an open Eulerian trail, then the two vertices of odd degree must have the same label and not both being of degree 3.

The correspondence between an Eulerian trail of a lattice diagram \( D(n, k) \) and a Hamiltonian cycle of the generalized Petersen graph \( P(n, k) \) is built by interpreting the edges in \( L(n, k) \). The interpretations of edges in \( L(n, k) \) are given as follows:

1. The vertical edge \((i, i \oplus k)\) in \( L(n, k) \) corresponds to an edge \((i', (i \oplus k)')\) in \( P(n, k) \).
2. The horizontal edge \((i, i \oplus 1)\) in \( L(n, k) \) corresponds to an edge \((i, i \oplus 1)\) in \( P(n, k) \).
3. Two edges in different directions incident to the vertex \( i \) of degree 2 in \( L(n, k) \) correspond to an edge \((i, i')\) in \( P(n, k) \).

A lattice diagram \((13, 4)\) for a generalized Petersen \( P(13, 4) \), for example, is shown in Fig. 1. In this diagram, there is an Eulerian trail \((0, 1, 5, 9, 10, 6, 2, 3, 7, 11, 12, 8, 7, 6, 5, 4, 4', 0', 0)\) in \( P(13, 4) \).

It is not difficult to see that an Eulerian trail in \( L(n, k) \) corresponds to a Hamiltonian cycle of \( P(n, k) \) and any Hamiltonian cycle of \( P(n, k) \) can be converted into an Eulerian trail in \( D(n, k) \). Hence, finding a Hamiltonian cycle of a generalized Petersen graph \( P(n, k) \) is finding an appropriate lattice diagram for \( P(n, k) \) [2].

In addition, Alspach et al. [2] proposed an amalgamating mechanism to generate lattice diagrams with various sizes. For example, we have a lattice diagram \( D(4k, k) \) in Fig. 2(a) and a lattice diagram \( D(4k + 2, k) \) in Fig. 2(b). By identifying the vertex with label \((4k - 1)\) of \( D(4k, k) \) and the vertex with label 0 of \( D(4k + 2, k) \) and relabeling all vertices of \( D(4k + 2, k) \) by adding \((4k - 1)\) to all the labels, we obtain a lattice diagram \( D(8k + 1, k) \) shown in Fig. 2(c). Thus, \( P(8k + 1, k) \) is Hamiltonian. With this mechanism, we can amalgamate \( r \) copies of \( D(4k, k) \) and \( s \) copies of \( D(4k + 2, k) \), where \( r \geq 1 \) and \( s \geq 0 \), to make a lattice diagram \( D(4k + (r - 1)(4k - 1) + s(4k + 1), k) \). Hence, \( P(4k + (r - 1)(4k - 1) + s(4k + 1), k) \) is Hamiltonian for \( r \geq 1 \) and \( s \geq 0 \).
Fig. 1. (a) A $D(13, 4)$ and (b) its corresponding Hamiltonian cycle in $P(13, 4)$.

Fig. 2. (a) A $D(4k, k)$, (b) a $D(4k + 2, k)$, and (c) a $D(8k + 1, k)$.

2. Preliminaries

In the section, some preliminaries about hyper-Hamiltonian graphs are introduced. A bipartite graph has no odd cycles. Hence, we have the following lemma.

**Lemma 1.** Any bipartite graph is not hyper-Hamiltonian.

By the definition of the generalized Petersen graphs, the next lemma is obtained.

**Lemma 2.** $P(n, k)$ is bipartite if and only if $n$ is even and $k$ is odd. Thus, $P(n, k)$ is not hyper-Hamiltonian if $n$ is even and $k$ is odd.

With **Lemma 2**, we may ask if $P(n, k)$ is not hyper-Hamiltonian if and only if $n$ is even and $k$ is odd. However, the statement is not true because of the following theorem.
Theorem 2 ([7]). \( P(n, 1) \) is hyper-Hamiltonian if and only if \( n \) is odd and \( P(n, 2) \) is hyper-Hamiltonian if and only if \( n \equiv 1, 3 \pmod{6} \).

Thus, \( P(n, 2) \) with \( n \) being even is neither bipartite nor hyper-Hamiltonian. However, we desire to know if there are other generalized Petersen graphs that are not hyper-Hamiltonian.

Let us consider the generalized Petersen graph \( P(12, 4) \) shown in Fig. 3(a). Suppose that \( P(12, 4) \) is hyper-Hamiltonian. Then \( P(12, 4) - 1 \) is Hamiltonian. We note that the vertex set \( \{0', 4', 8'\} \) induces a complete graph \( K_3 \). Any Hamiltonian cycle in \( P(12, 4) - 1 \) must traverse the vertex set \( \{0', 4', 8'\} \) consecutively because \( P(12, 4) \) is a cubic graph. Similarly, any Hamiltonian cycle traverses the vertex sets \( \{1', 5', 9'\}, \{2', 6', 10'\}, \) and \( \{3', 7', 11'\} \) consecutively. For this reason, we can define a graph \( P'(12, 4) \) obtained from \( P(12, 4) \) by shrinking the vertices \( 0', 4', \) and \( 8' \) into a vertex \( 0 \), shrinking the vertices \( 1', 5', \) and \( 9' \) into a vertex \( 1 \), shrinking the vertices \( 2', 6', \) and \( 10' \) into a vertex \( 2 \) and shrinking the vertices \( 3', 7', \) and \( 11' \) into a vertex \( 3 \) as shown in Fig. 3(b). Thus, \( P(12, 4) - 1 \) is Hamiltonian if and only if \( P'(12, 4) - 1 \) is Hamiltonian. However, \( P'(12, 4) \) is a bipartite graph with 8 vertices in each partite set. Therefore, \( P'(12, 4) - 1 \) is not Hamiltonian. We get a contradiction. Thus, \( P(12, 4) - 1 \) is not Hamiltonian.

The lattice model is a useful tool to solve the Hamiltonianity on generalized Petersen graphs. Now, we slightly modify the model to show that many generalized Petersen graphs are hyper Hamiltonian.

First, we give examples to demonstrate the variation. In Fig. 4(a), an open Eulerian trail \( \{0, 1, 7, 8, 2, 3, 9, 10, 4, 5, 11, 12, 6, 0\} \) corresponds to the Hamiltonian cycle \( \{0, 1, 1', 7', 7, 8, 8', 2', 2, 3, 3', 9', 9, 10, 10', 4', 4, 5, 5', 11', 11, 12, 12', 6', 0'\} \) of \( P(13, 6) - 6 \) using the same interpretation described above. The reason that the open Eulerian trail corresponds to the Hamiltonian cycle of \( P(13, 6) - 6 \) is that the vertex with label 6 is traversed only in a vertical direction and all the other vertices are traversed in both directions. Again, the open Eulerian trail in Fig. 4(b), \( \{0, 1, 2, 8, 9, 3, 4, 10, 11, 5, 6, 7, 13, 12, 6, 0\} \) corresponds to the Hamiltonian cycle \( \{0, 1, 2, 8', 8, 9, 9', 3', 3, 4, 4', 10', 10, 11, 11', 5', 5, 6, 7, 7', 13', 13, 12, 12', 6', 0', 0\} \) in \( P(14, 6) - 1' \). In this diagram, the vertex with label 1 is traversed only in a horizontal direction and all the other vertices are traversed in both directions.

In other words, finding a lattice diagram \( D^h_i(n, k) \) with an Eulerian trail such that the vertex with label \( i \) is traversed only in a vertical direction and all the other vertices are traversed in both directions is finding a Hamiltonian cycle of \( P(n, k) - i \). Similarly, finding a lattice diagram \( D^v_i(n, k) \) with an Eulerian trail such that the vertex with label
3. The generalized Petersen graph \( P(3k, k) \)

In this section, we prove that \( P(3k, k) \) is hyper-Hamiltonian if and only if \( k \) is odd. We first prove that \( P(3k, k) \) is not hyper-Hamiltonian if \( k \) is even. The proof is similar to the proof of the fact that \( P(12, 4) \) is not hyper-Hamiltonian in Section 2.

Let \( k \) be an even integer. Suppose that \( P(3k, k) \) is hyper-Hamiltonian. Then, there exists a Hamiltonian cycle of \( P(3k, k) - 1 \). By the definition of the generalized Petersen graph, the set \( \{i', (i+k)', (i+2k)\}' \) induces a cycle of length 3 for \( 0 \leq i \leq k - 1 \). Therefore, any Hamiltonian cycle in \( P(3k, k) - 1 \) will traverse the vertex set \( \{i', (i+k)', (i+2k)\}' \) consecutively for \( 0 \leq i \leq k - 1 \). Therefore, vertices \( i', (i+k)', \) and \( (i+2k)' \) can be regarded as a vertex for
0 ≤ i ≤ k − 1. Let \( P'(3k, k) \) be the graph obtained from \( P(3k, k) \) by shrinking the vertices \( i', (i + k)', (i + 2k)' \) into a new vertex, say \( i \), for 0 ≤ i ≤ k − 1. It is not difficult to see that \( P(3k, k) \) is Hamiltonian if and only if \( P'(3k, k) \) is Hamiltonian.

Now, we claim that \( P'(3k, k) \) is a bipartite graph. Let \( X = \{1, 3, 5, \ldots, (3k - 1)\} \) \( \cup \{i \} \mid 0 ≤ i ≤ k - 1 \) and \( Y = \{0, 2, 4, 6, \ldots, (3k - 2)\} \) \( \cup \{i \} \mid 0 ≤ i ≤ k - 1 \) and \( i \) is odd). It is easy to check that \( (X, Y) \) forms a bipartition of \( P(3k, k) \) with \( |X| = |Y| = 2k \). Hence, \( P'(3k, k) \) is a bipartite graph. By Lemma 1, \( P'(3k, k) \) is not Hamiltonian, and hence, \( P(3k, k) \) is not Hamiltonian. Therefore, \( P(3k, k) \) is not hyper-Hamiltonian if \( k \) is even.

Thus, we consider that \( k \) is odd. Suppose \( n = 1 \). Obviously, \( (1, 1', 0', 2', 2, 1) \) forms a Hamiltonian cycle for \( P(3, 1) - 0 \) and \( (0, 1, 1', 2', 2, 0) \) forms a Hamiltonian cycle for \( P(3, 1) - 0'. Thus, \( P(3, 1) \) is hyper-Hamiltonian.

Suppose \( k = 3 \). Then \( (1, 2, 3, 3', 0', 6', 6, 7, 8, 8', 2', 5', 5, 4, 4', 7', 1', 1) \) forms a Hamiltonian cycle of \( P(9, 3) - 0 \) and \( (0, 1, 2, 3, 3', 6', 6, 7, 7', 1', 4', 5, 5', 2', 8', 8, 0) \) forms a Hamiltonian cycle for \( P(9, 3) - 0'. Thus, \( P(9, 3) \) is hyper-Hamiltonian.

Suppose \( k ≥ 5 \). A lattice diagram \( D_{2k-3}(3k, k) \) with \( k \equiv 1 \pmod{4} \) is shown in Fig. 5(a) and a lattice diagram \( D_{2k-3}(3k, k) \) with \( k \equiv 3 \pmod{4} \) is shown in Fig. 5(b). Moreover, a lattice diagram \( D_{k-1}^h(3k, k) \) is shown in Fig. 5(c). Thus, \( P(3k, k) \) is hyper-Hamiltonian. Thus, we have the following theorem.

**Theorem 3.** \( P(3k, k) \) is hyper-Hamiltonian if and only if \( k \) is odd.

4. The generalized Petersen graph \( P(n, 3) \)

**Theorem 4.** The generalized Petersen graph \( P(n, 3) \) is hyper-Hamiltonian if and only if \( n \) is odd.

**Proof.** By the definition of \( P(n, k) \), we consider \( n ≥ 7 \). By Lemma 2, \( P(n, 3) \) is not hyper-Hamiltonian if \( n \) is even. Thus, we consider the case \( n \) is odd.

Suppose \( n = 7 \). Then \( (0, 0', 4', 1', 1, 2, 3, 3', 6', 2', 5', 5, 6, 0) \) is a Hamiltonian cycle for \( P(7, 3) - 4 \) and \( (0, 1, 2, 3, 4, 4', 0', 3', 6', 2', 5', 5, 6, 0) \) is a Hamiltonian cycle for \( P(7, 3) - 1' \). Thus, \( P(7, 3) \) is hyper-Hamiltonian.

Suppose \( n = 9 \). By Theorem 3, \( P(9, 3) \) is hyper-Hamiltonian in Theorem 3.

Suppose \( n ≥ 11 \). A lattice diagram \( D_4^3(n, 3) \) is shown in Fig. 6(a) and a lattice diagram \( D_h^4(n, 3) \) is shown in Fig. 6(b). Thus, \( p(n, 3) \) is hyper-Hamiltonian.

Therefore, \( P(n, 3) \) is hyper-Hamiltonian if and only if \( n \) is odd. \( \Box \)

5. The generalized Petersen graph \( P(n, 4) \)

**Theorem 5.** The generalized Petersen graph \( P(n, 4) \) is hyper-Hamiltonian if and only if \( n ≠ 12 \).
Theorem 3. (e), (f), (g), and (h), we have four lattice diagrams (a), (b), (c), and (d), we have four lattice diagrams

By the definition of $P(n, k)$, we consider the cases $n \geq 9$. With Theorem 3, $P(12, 4)$ is not hyper-Hamiltonian. Thus, we consider the cases $n \geq 9$ and $n \neq 12$.

Suppose $n = 9$. Then $\langle 0, 1, 2, 2', 7', 7, 6, 6', 1', 5', 5, 4, 3, 3', 8', 4', 0', 0 \rangle$ is a Hamiltonian cycle for $P(9, 4) - 8$ and $\langle 0, 1, 1', 5', 0', 4', 8', 3', 7', 2', 2, 3, 4, 5, 6, 7, 8, 0 \rangle$ is a Hamiltonian cycle for $P(9, 4) - 6'$. Thus, $P(9, 4)$ is hyper-Hamiltonian.

Suppose $n = 10$. Then $\langle 0, 1, 2, 3, 3', 9', 5', 1', 7', 7, 8, 8', 2', 6', 6, 5, 4, 4', 0', 0 \rangle$ is a Hamiltonian cycle for $P(10, 4) - 9$ and $\langle 0, 1, 1', 5', 5, 4, 4', 8', 2', 2, 3, 3', 9', 9, 8, 7, 6, 6', 0', 0 \rangle$ is a Hamiltonian cycle for $P(10, 4) - 7'$. Thus, $P(10, 4)$ is hyper-Hamiltonian.

Suppose $n = 11$. Then $\langle 0, 1, 1', 8', 8, 9, 9', 5', 5, 6, 7, 7', 3', 10', 6', 2', 2, 3, 4, 4', 0', 0 \rangle$ is a Hamiltonian cycle for $P(11, 4) - 10$ and $\langle 0, 1, 1', 5', 9', 2', 2, 3, 3', 10', 6', 6, 5, 4, 4', 0', 7', 7, 8, 9, 10, 0 \rangle$ is a Hamiltonian cycle for $P(11, 4) - 8'$.

Suppose $n \geq 13$. In Fig. 7(a), (b), (c), and (d), we have four lattice diagrams $D^v_{n-1}(n, 4)$ depending on the value of $n \pmod{4}$. In Fig. 7(e), (f), (g), and (h), we have four lattice diagrams $D^h_{n-3}(n, 4)$ depending on the value of $n \pmod{4}$.

Thus, $P(n, 4)$ is hyper-Hamiltonian if and only if $n \neq 12$. □
6. The generalized Petersen graph \( P(n, k) \) with \( k \geq 5 \)

With Theorems 2, 4 and 5, we can recognize those hyper-Hamiltonian Petersen graphs \( P(n, k) \) with \( k \leq 4 \). Now, we consider the case \( k \geq 5 \). By Lemma 2, \( P(n, k) \) is not hyper-Hamiltonian if \( n \) is even and \( k \) is odd. In this section, we will prove that \( P(n, k) \) is hyper-Hamiltonian if (1) \( k \geq 5 \) is odd and \( n \) is odd with \( n \) sufficiently large with respect to \( k \); and (2) \( k \) is even and \( k \geq 6 \) and \( n \) is sufficiently large with respect to \( k \). We will use the amalgamating mechanism, proposed by Alspach et al. [2], described in Section 2 to obtain suitable lattice diagrams.

We first consider the case that \( k \) is even with \( k \geq 6 \). We will use four basic lattice diagrams. Let \( D(4k, k) \) and \( D(4k + 2, k) \) be the lattice diagrams shown in Fig. 2(a) and Fig. 2(b), respectively. Moreover, let \( D'_{k}(2k + 1, k) \) be the lattice diagram shown in Fig. 8(a) and \( D'_{k}(2k + 2, k) \) be the lattice diagram shown in Fig. 8(b). For illustration, we amalgamate a \( D_{6}^{v}(13, 6) \) and a \( D(26, 6) \) to obtain a \( D_{6}^{v}(38, 6) \) in Fig. 8(c). Note that \( 38 = 13 + 26 - 1 \) because a vertex is duplicated during the amalgamating mechanism. Similarly, we amalgamate a \( D_{6}^{v}(14, 6) \) and a \( D(24, 6) \) to obtain a \( D_{6}^{v}(37, 6) \) in Fig. 8(c).

Assume that \( n \geq 2k + 2 + (4k - 1)(4k + 1) \). Since \( \gcd(4k - 1, 4k + 1) = 1 \), there exist nonnegative integers \( r \) and \( s \) such that \( n = 2k + 1 + r(4k - 1) + s(4k + 1) \). Similarly, there exist nonnegative integers \( t \) and \( u \) such that \( n = 2k + 2 + t(4k - 1) + u(4k + 1) \). We can amalgamate one copy of \( D'_{k}(2k + 1, k) \), \( r \) copies of \( D(4k, k) \), and \( s \) copies of \( D(4k + 2, k) \) to make a lattice diagram \( D'_{k}(n, k) \). Thus, \( P(n, k) - k \) is Hamiltonian. Again, we can amalgamate one copy of \( D'_{k}(2k + 2, k) \), \( t \) copies of \( D(4k, k) \), and \( u \) copies of \( D(4k + 2, k) \) to make a lattice diagram \( D'_{k}(n, k) \). Thus, \( P(n, k) - 1' \) is Hamiltonian.
Thus, we have the following theorem.

**Theorem 6.** Suppose that $k$ is even with $k \geq 6$. Then $P(n, k)$ is hyper-Hamiltonian if $n \geq 2k + 2 + (4k - 1)(4k + 1)$.

Now, we consider both $k$ and $n$ to be odd integers with $k \geq 5$. We will use four basic lattice diagrams. Let $D_v^0(6k - 3, k)$ be the lattice diagram shown in Fig. 9(a), $D_h^1(2k + 1, k)$ be the lattice diagram shown in Fig. 9(b), $D(2k, k)$ be the lattice diagram in Fig. 9(c), and $D(6k - 2, k)$ be the lattice diagram in Fig. 9(d). For illustration, we amalgamate a $D^0_h(27, 5)$ and a $D^1(10, 5)$ to obtain a $D^0_h(37, 5)$ in Fig. 9(e). We observe that $37 = 27 + 10$ which is
different from the previous case. The two vertices labeled with 0 in $D^h_0(27, 5)$ are the terminal vertices of the open Eulerian trail. We identify one of the vertices labeled with 0 in $D^h_0(27, 5)$ with one of the vertices labeled with 0 in $D(10, 5)$ to obtain a $D^h_0(37, 5)$. Similarly, we amalgamate a $D^h_0(11, 5)$ and a $D(28, 5)$ to obtain a $D^h_0(39, 5)$ in Fig. 8(f).

Assume that $n \geq 6k - 3 + (2k)(6k - 2)$. Since $\gcd(2k, 6k - 2) = 2$, there exist nonnegative integers $r$ and $s$ such that $n = 6k - 3 + r(2k) + s(6k - 2)$. Moreover, there exist nonnegative integers $t$ and $u$ such that $n = 6k - 3 + t(2k) + u(6k - 2)$. We can amalgamate one copy of $D^h_0(6k - 3, k)$, $r$ copies of $D(2k, k)$, and $s$ copies of $D(6k - 2, k)$ to make a lattice diagram $D^h_0(n, k)$. Similarly, we can amalgamate one copy of $D^h_0(2k + 1, k)$, $t$ copies of $D(2k, k)$, and $u$ copies of $D(6k - 2, k)$ to make a lattice diagram $D^h_0(n, k)$. Thus, $P(n, k)$ is hyper-Hamiltonian.

**Theorem 7.** Assume that $k$ and $n$ are odd integers with $k \geq 5$. Then $P(n, k)$ is hyper-Hamiltonian if $n \geq 6k - 3 + (2k)(6k - 2)$.

7. Concluding remark

In this paper, some necessary conditions and some sufficient conditions for the hyper-Hamiltonian generalized Petersen graphs are proved. Moreover, we show that $P(n, k)$ is hyper-Hamiltonian if $k \geq 5$ and $n$ sufficiently large. More precisely, we proved that (1) $P(3k, k)$ is hyper-Hamiltonian if and only if $k$ is odd; (2) $P(n, 3)$ is hyper-Hamiltonian if and only if $n$ is odd; (3) $P(n, 4)$ is hyper-Hamiltonian if and only if $n \neq 12$; (4) $P(n, k)$ is hyper-Hamiltonian if $n \geq 2k + 2 + (4k - 1)(4k + 1)$ and $k$ is even with $k \geq 6$; and (5) $P(n, k)$ is hyper-Hamiltonian if $k$ is odd with $k \geq 5$ and $n$ is odd with $n \geq 6k - 3 + 2k(6k - 2)$.

The first contribution toward the hyper-Hamiltonian generalized Petersen graphs was made by Albert et al. [7]. They proved that $P(n, 1)$ is hyper-Hamiltonian if and only if $n$ is odd and $P(n, 2)$ is hyper-Hamiltonian if and only if $n \equiv 1, 3 \pmod{6}$.

In future works, we will be considering the classification of hyper-Hamiltonian generalized Petersen graphs. That is, we want to show that all nonbipartite generalized Petersen graphs except for $\{P(3k, k) \mid k \text{ is even}\} \cup \{P(n, 2) \mid n \equiv 0, 2, 4, 5 \pmod{6}\}$ are hyper-Hamiltonian.

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**References**


